GEOMETRY FINAL EXAM

Attempt all questions. Two of the four questions below have several parts. You might be able to do a later part by using a part which came before. Also you might be able to do a later part without knowing how to do a previous part (and by just assuming it). Total marks 50

(1) (11 marks) Given any α, β, γ such that $0 < \alpha + \beta + \gamma < \pi$, show that there is a hyperbolic triangle with angles α, β, γ.

(2) (11 marks) Prove that every isometry of S^2 is a composition of at most three reflections in great circles.

(3) (4+4+4+2 = 14 marks) Let $S^2 \subset \mathbb{R}^3$ be the unit sphere $\{x^2 + y^2 + z^2 = 1\}$. Consider the strict spherical triangle ABC with vertices $A = (0, 0, 1)$, $B = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, $C = (0, \frac{\sqrt{3}}{2}, \frac{1}{2})$.with sides a, b, c (the side a is opposite the vertex A etc) and angles α, β, γ (the angle α is at the vertex A etc). Let $A'B'C'$ be the dual triangle of ABC with angles α', β', γ' and sides a', b', c' (same conventions for naming sides and angles as before). Some of your answers can be in terms of inverses of trigonometric functions.

- (a) Find a, b, c and α, β, γ.
- (b) Using the method to draw dual triangles, find out the coordinates of the points A', B', C'.
- (c) Find a', b', c' and α', β', γ' (without using the fact that $A'B'C'$ is dual to ABC, that is just as you did for the triangle ABC in part(a)).
- (d) Using the definition of dual triangles, verify your results of part(c).

(4) (4+5+3+2 = 14 marks) Let $\mathcal{D}$ be the open unit disc $\{x^2 + y^2 < 1\}$ (the Poincaré disc model of hyperbolic geometry) and let $\mathcal{C} = \{x^2 + y^2 = 1\}$ be its boundary.

- (a) Let $P = (a, b) \in \mathcal{D}$ be a point which is not the origin $O = (0, 0)$. Let $r > 0$ be a fixed real number. Let l be a d-line which is part of a circle C of radius r and such that $P \in l$. Prove that there are at most two such d-lines l. Is there some transformation of the plane which interchanges the two circles (in the case when there are two d-lines and two circles)?
- (b) Consider a hyperbolic triangle ABC with sides $AB = l_1, BC = l_2$ and $AC = l_3$ which are all segments of d-lines (hence part of three circles C_1, C_2 and C_3). Suppose the angle bisectors of the angles of the triangle ABC at A and at B meet at the origin $O = (0, 0)$. Prove that the three circles C_1, C_2, C_3 all have the same radii.
- (c) Prove that the bisector of the angle of the triangle ABC at C also passes through O.
- (d) Write the statement of the general theorem which you have thus proved.